Proofs and exercises

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Contents

1 General definitions and remarks

Definition 1.1. Let $\mathbb{N}$ denote the set of positive natural numbers.

Definition 1.2. If A, B and $A_0 \subseteq A$ are sets, and a function $f : A \to B$, then $f|A_0$ denotes the function f whose domain has been restricted to A_0, such that $\{f(a) \,|\, a \in A_0\}$.

Definition 1.3. A section S_n of the positive integers, is a set that equals $\{x \in \mathbb{N} \,|\, x < n\}$. A section S_{n+1} denotes the set $\{x \in \mathbb{N} \,|\, x \leq n\}$.

2 Set theory and logic

2.1 Functions

Lemma 2.1. *Let $f : A \to B$. If there are functions $g : B \to A$ and $h : B \to A$ such that $g(f(a)) = a$ for every $a \in A$. and $f(h(b)) = b$ for every $b \in B$, then f is bijective and $g = h = f^{-1}$.*

Proof. In order to prove that f is bijective, one must to prove that f is surjective and injective. To show that f is surjective; we proceed by contradiction. Assume that f is not surjective, then there is at least one element $b_0 \in B$ for which there is no corresponding element $a \in A$. We have a function $f \circ h : B \to A \to B$ that is defined by $(f \circ h)(b) = b$ for *all* $b \in B$, contradicting our claim that the map $f : A \to B$ is not surjective, since in that case $f(a) \neq b_0$ for all $a \in A$ and not satisfy the condition that $(f \circ h)(b) = b$ for all $b \in B$.

To show that f is injective, consider a function f such that $f(a) = f(a')$, since, by the lemma, we have an function g such that $g(f(x)) = x$ for all $x \in A$, then we apply g to the equation above and get

$$f(a) = f(a') \Rightarrow g(f(a)) = g(f(a')) \Rightarrow a = a',$$

thus meeting the definition of injectivity. f is then bijective as desired.

To prove that $f^{-1} = h = g$. Since $f : A \to B$ is bijective, such that $f : a_n \mapsto b_k$, there is a bijective function $f^{-1} : B \to A$, such that $f^{-1} : b_k \mapsto a_n$. We define a function $f \circ f^{-1} : B \to A \to B$, it follows that $f \circ f^{-1} : b_k \mapsto a_n \mapsto b_k$ so that $(f \circ f^{-1})(b) = b$ for all $b \in B$, since $(f \circ h)(b) = b$ for all $b \in B$, we conclude that $h : b_k \mapsto a_n$, and so $h = f^{-1}$.

Lets consider a function $f^{-1} \circ f : A \to B \to A$, by our previous definition of f and f^{-1} it follows that $f^{-1} \circ f : a_n \mapsto b_k \mapsto a_n$. So $(f^{-1} \circ f)(a) = a$ for all $a \in A$. Since $(g \circ f)(a) = a$ for all $a \in A$, we conclude that $g : b_k \mapsto a_n$ then $f^{-1} = g = h$ as desired. □

2.1.1 Exercises

Exercise 2.1. Let $f : A \to B$. Let $A_0 \subseteq A$ and $B_0 \subseteq B$.

1. Show that $A_0 \subseteq f^{-1}(f(A_0))$ and that equality holds if f is injective.

2. Show that $f(f^{-1}(B_0)) \subseteq B_0$ and that equality holds if f is surjective.

Solution 2.1. For the first question. Let the set $f(A_0)$ be denoted by B_A. We will show that A_0 is contained in the preimage of B_A under f. If f is not injective, there may be a map to some elements $a_n \mapsto b_k$ and $a_m \mapsto b_k$ for $b_k \in B_A$ for some $a_n \neq a_m$. If $a_n \in A_0$ then a_m may or may not be in A_0. It follows that the preimage of B_A under f is at least equal to A_0, and thus $A_0 \subseteq f^{-1}(f(A_0))$. If f injective, at most one unique element $a \in A$ has been mapped to each element $b \in B$, then the preimage of B_A under f equals A_0.

For the second question. Let the set $f^{-1}(B_0)$ be denoted by A_B. We will show that the image of A_B under f is a subset of B_0. If f is not surjective, the set B_0, is at least equal to the set $f(A_B)$, as f may or may not map an element $a \in A_B$ to every element $b \in B_0$. And so $f(f^{-1}(B_0)) \subseteq B_0$. If f is surjective, then it is a given that all $a \in A_B$ is mapped to B_0, and thus $f(f^{-1}(B_0)) = B_0$.

Exercise 2.2. Let $f : A \to B$ and let $A_i \subseteq A$ and $B_i \subseteq B$ for $i \in \{0, 1\}$. Show that the follwing is true:

1. $B_0 \subseteq B_1 \Rightarrow f^{-1}(B_0) \subseteq f^{-1}(B_1)$.
2. $f^{-1}(B_0 \cup B_1) = f^{-1}(B_0) \cup f^{-1}(B_1)$.
3. $f^{-1}(B_0 \cap B_1) = f^{-1}(B_0) \cap f^{-1}(B_1)$.
4. $f^{-1}(B_0 \setminus B_1) = f^{-1}(B_0) \setminus f^{-1}(B_1)$.
5. $A_0 \subseteq A_1 \Rightarrow f(A_0) \subseteq f(A_1)$.
6. $f(A_0 \cup A_1) = f(A_0) \cup A(A_1)$.
7. $f(A_0 \cap A_1) \subseteq f(A_0) \cap f(A_1)$. Show that equality holds if f is injective.
8. $f(A_0 \setminus A_1) \supseteq f(A_0) \setminus f(A_1)$. Show that equality holds if f is injective.

Solution 2.2. One. Consider a function $f : A \to B$ such that $f : a_n \mapsto b_k$, if the element $b_k \in B_1$ but not in B_0, then $f^{-1}(B_0) = \varnothing$, and the equation holds, likewise if it is in B_0, then $a_n \in f^{-1}(B_0)$, and be in $f^{-1}(B_1)$ aswell, as any element of the image that is in B_0, will be in B_1. So, every element in the preimage of B_1 under f will either be an element of $f^{-1}(B_0)$ or not—there is no case in which there will be more elements in $f^{-1}(B_0)$ than in $f^{-1}(B_1)$. So the equation holds.

2.2 Principle of recursive definition

Lemma 2.2. *There exists a function $f : \{1, \ldots, n\} \to C$ that satisfies* (1) *for all i in its domain.*

Proof. By induction consider a function $f : \{1, \ldots, n\} \to A$, where $a_0 \in A$, defined by the equation $f(1) = a_0$. It is clear that it satisfies (1) for $n = 1$. By inductive hypothesis the lemma is true for $n - 1$. To prove it for n let there be a function $f' : \{1, \ldots, n-1\} \to A$ that satisfies (1). Let f' be defined by $f'(i) = f(i)$ for $i \in \{1, \ldots, n-1\}$, and f by

$$f(1) = a_0$$
$$f(n) = \rho(f'|\{1, \ldots, n-1\}) \qquad \text{for } n > 1$$

Now, since $f'(i) = f(i)$ for $i \in \{1, \ldots, n-1\}$, f satisfies (1). $\square$

Lemma 2.3. *If two functions $f : \{1, \dots, n\} \to C$ and $g : \{1, \dots, m\} \to C$ satisfy* (1)*, then $f(i) = g(i)$ for all i in both domains.*

Proof. In Lemma 2.2 we showed that there exists a function $h : \{1, \dots, n\} \to C$ that satisfy (1). Let $f : \{1, \dots, n\} \to A$ and $g : \{1, \dots, m\} \to A$ be two such functions. By contradiction, let i be the smallest number for which $g(i) \neq f(i)$, it is not 1 since $f(1) = g(1) = a_0$ by definition. So we have that $f(j) = g(j)$ for all $j < i$. Since g and f satisfy (1), it follows that $f(\{1, \dots, i-1\}) = g(\{1, \dots, i-1\})$. Then $f(i) = g(i)$ for all i in both of their domains, contrary to our choice of i. □

Theorem 2.4 (Principle of recursive definition)**.** *Let A be a set and $a_0 \in A$. Then consider a function ρ that assigns to each function f mapping a nonempty section of $\mathbb{N}$ to A, an element of A. Then there is a unique function*

$$h : \mathbb{N} \to A$$

That satisfies the equation

$$\begin{aligned} h(1) &= a_0 \\ h(n) &= \rho(h|\{1, \dots, n-1\}) \qquad \text{for } i > 1. \end{aligned} \tag{1}$$

Proof. Consider a function $f_n : \{1, \dots, n\} \to C$, that satisfy (1), whose existence we have proven proven in Lemma 2.2. Let the function $h : \mathbb{N} \to C$ be defined to be the union U of all the rules that define f_n. The rules for f_n is a subset of $\{1, \dots, n\} \times C$, then U is a subset of $\mathbb{N} \times C$, and as such can be represented as ordered pairs in the form $(i, f_n(i))$ for $i \leq n$. Now it is clear that since $h(i) = f_n(i)$, and that f satisfies (1) for all i is its domain, so does h. By Lemma 2.3, $f_n(i) = f_m(i)$ for $i \leq m, n$, it follows that there is only one element of U for each i, $h(i)$ is thus well-defined.

Since there exists a function $h : \mathbb{N} \to C$ that satisfy (1), to prove that it is unique, consider by contradiction two functions $h_a : \mathbb{N} \to C$ and $h_b : \mathbb{N} \to C$ that satisfy (1) for all $i \in \mathbb{N}$, such that $h_a(i) \neq h_b(i)$. We have already proven in Lemma 2.3 that two functions $\{1, \dots, n\} \to C$ and $\{1, \dots, m\} \to C$ that satisfy (1) are equal for all i in both of their domains. Since h_a and h_b have $\mathbb{N}$ as their domain and they satisfy (1), $h_a(i) = h_b(i)$ for all $i \in \mathbb{N}$, contradicting our statement that $h_a(i) \neq h_b(i)$. Then h is unique as desired. □

2.2.1 Exercises

Exercise 2.3. Let $(b_1, b_2, \ldots)$ be an infinite sequence of real numbers. The sum $\sum_{k=1}^{n} b_k$ is defined by induction as follows:

$$\sum_{k=1}^{n} b_k = b_1 \qquad \text{for } n = 1$$
$$\sum_{k=1}^{n} b_k = \left(\sum_{k=1}^{n-1} b_k\right) + b_n \qquad \text{for } n > 1.$$

Choose ρ so that Lemma 2.4 applies to define the sum rigorously.

Solution 2.3. We wish to to apply Theorem 2.4 to define a function $h : \mathbb{N} \to \mathbb{R}$ rigorously such that $h(n) = \sum_{k=1}^{n} b_k$. To apply this Theorem, let ρ be defined by the equation $\rho(f) = f(m) + b_{m+1}$, where $f : \{1, \ldots, m\} \to \mathbb{R}$. Then there exists a function $h : \mathbb{N} \to \mathbb{R}$ such that

$$h(1) = b_1$$
$$h(n) = \rho(h|\{1, \ldots, i-1\}) \qquad \text{for } i > 1$$

So this means that $h(1) = b_1$ and $h(i) = h(i-1) + b_i$ for $i > 1$. If we denote $h(n)$ by $\sum_{k=1}^{n} b_k$, we have

$$\sum_{k=1}^{1} b_k = b_1$$
$$\sum_{k=1}^{n} b_k = \left(\sum_{k=1}^{n-1} b_k\right) + b_n \qquad \text{for } n > 1.$$

as desired.

Exercise 2.4. Let $(b_1, b_2, \ldots)$ be an infinite sequence of real numbers. The product $\prod_{k=1}^{n} b_k$ is defined by the equations

$$\prod_{k=1}^{1} b_k = b_1 \qquad \text{for } n = 1$$
$$\prod_{k=1}^{n} b_k = \left(\prod_{k=1}^{n-1} b_k\right) \cdot b_n \qquad \text{for } n > 1$$

Use Theorem 2.4 to define this product rigorously.

Solution 2.4. We wish to apply Theorem 2.4 to define a function $h : \mathbb{N} \to \mathbb{R}$ rigorously so that $h(i) = \prod_{k=1}^{i} b_k$. To apply the Theorem, let ρ be defined by the equation $\rho(f) = f(m) \cdot b_{m+1}$, where $f : \{1, \ldots, m\} \to \mathbb{R}$. Then there exists a unique function

$$\begin{aligned} h(1) &= b_1 \\ h(n) &= \rho(h|\{1, \ldots, n-1\}) \qquad \text{for } n > 1. \end{aligned}$$

Then we have $h(1) = b_1$ and $h(n) = h(n-1) \cdot b_n$ for $n > 1$. Let $h(n)$ be denoted by $\prod_{k=1}^{n}$, then we have

$$\begin{aligned} \prod_{k=1}^{1} b_k &= b_1 \qquad \text{for } n = 1 \\ \prod_{k=1}^{n} b_k &= \left(\prod_{k=1}^{n-1} b_k \right) \cdot b_n \qquad \text{for } n > 1 \end{aligned}$$

as desired.

Exercise 2.5. Obtain a definition of $n!$ for $n \in \mathbb{N}$ as special case of Exercise 2.4.

Solution 2.5. We wish to define $h(n) = \prod_{k=1}^{n} k = n!$ as a special case of Excercise 2.4. We restrict the codomain of h to $\mathbb{N}$. If we let b_{m+1} be denoted by $m+1$, b_1 by 1, and b_k by k, we can denote $h(n)$ by $n!$ since

$$\begin{aligned} 1! &= \prod_{k=1}^{1} k = 1 \qquad \text{for } n = 1 \\ n! &= \prod_{k=1}^{n} k = \left(\prod_{k=1}^{n-1} k \right) \cdot n \qquad \text{for } n > 1 \end{aligned}$$

as desired.

Exercise 2.6. Obtain a definition of a^n as a special case of Excercise 2.4.

Solution 2.6. By applying the definition of $\prod_{k=1}^{n} b_k$ we got in Exercise 2.4 we have $h(n) = h(n-1) \cdot a$, by letting $a = a_k = b_k$. and if we denote $h(i)$ by a^i we have

$$\begin{aligned} a^1 &= \prod_{k=1}^{1} a_k = a \qquad \text{for } n = 1 \\ a^n &= \prod_{k=1}^{n} a_k = \left(\prod_{k=1}^{n-1} a_k \right) \cdot a = a^{n-1} \cdot a \qquad \text{for } n > 1 \end{aligned}$$

as desired.