1 Some basic proofs

Definition 1. We say that p is an even integer, if there exists an integer q that satisfies $2q = p$.

Definition 2. We say that an integer p is not even, when there exists no ingteger q such that $2q = p$. Since $p = 2q + 1$ satisfies this condition, it is not even.

Definition 3. We say that an integer r is a multiple of and integer p if there exists an integer q such that $pq = r$.

Proposition 1. *The square of an even integer is even.*

Proof. Let a be an even number, by definition 1 there exists an integer q that satisfies $a = 2q$, it follows that $a^2 = 2aq$. Since a^2 equals twice aq and is thrus even, we have proved that the square of an even number is even. □

Proposition 2. *For any integer n if n^2 is odd then n is odd.*

Proof. By definition 2 $n^2 = (2a + 1)(2a + 1)$ is odd, which implies $n^2 = 2a(2a + 1) + (2a + 1)$, n^2 is thrus odd. Since $n^2 = (2a + 1)(2a + 1)$ then $n = 2a + 1$ and is thrus odd, as required. □

Proposition 3. *For real numbers a and b if $0 \leq a < b$ then $a^2 < b^2$.*

Proof. For $a > 0$ it follows that $a < b \Rightarrow a^2 < ab$ and $a < b \Rightarrow ab < b^2$, and hence $0 < a < b \Rightarrow a^2 < b^2$. For $a = 0$ it follows that $a = 0 \Rightarrow a^2 = 0$, and since $b > 0 \Rightarrow b^2 > 0$ it follows that $a^2 < b^2$. It is now proved that $0 \leq a < b \Rightarrow a^2 < b^2$. □

Corollary 1. *For all real numbers a and b if $|a| < |b|$ then $a^2 < b^2$.*

Proof. Since proposition 3 and $|a|^2 = a^2$ is true, it follows that if $|a| < |b|$ then $a^2 < b^2$. □

Proposition 4. *There are no integers m and n such that* $14m + 21n = 100$.

Proof. Suppose for contradiction that there exists integers m and n such that $14m + 21n = 100$ this implies $7(2m + 3n) = 100$.

By definition 3 that means if such integers exist, it would imply that there exists an integer q such that $7q = 100$. There exists no such integer. Thrus there is no integer m and n such that $14m - 21n = 100$. □

Proposition 5. *If* $a^2 \geq 7a$ *then* $a \geq 7$ *or* $a \leq 0$.

Proof. Suppose that $a \not\leq 0$ then $a^2 \cdot a^{-1} \geq 7a \cdot a^{-1}$ which equals $a \geq 7$. If $a \leq 0$ then $a^2 \geq 7a$ is true since a^2 is non-negative and $7a$ is non-positive. Thrus $a^2 \geq 7a$ then $a \geq 7$ or $a \leq 0$. □

2 Quantifiers

Proposition 6. *The statement* $\forall m, n \in \mathbb{Z}^+, n \geq m$ *is false.*

Proof. Consider for contradiction that $\exists m, n \in \mathbb{Z}^+, n \not\geq m$ is true if $m > 1$, it follows that $\forall m, n \in \mathbb{Z}^+, n \geq m$ is false. □

Proposition 7. *The statement* $\exists m, n \in \mathbb{Z}^+, m \leq n$ *is true.*

Proof. Considder for contradiction that if $m = n$ or $m < n$ then the satement $\forall m, n \in \mathbb{Z}^+, m \not\leq n$ is false, and $\exists m, n \in \mathbb{Z}^+, m \leq n$ is thrus true. □

Proposition 8. *The statement* $\forall m \in \mathbb{Z}^+, \exists n \in \mathbb{Z}^+, m \leq n$ *is false.*

Proof. Consider for contradiction $\exists m \in \mathbb{Z}^+, \forall n \in \mathbb{Z}^+, m \not\leq n$, since it is certainly true that there exists a positive integer $m = n + 1$ for all n. □