Proofs and exercises

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Contents

1 Definitions and remarks

Definition 1. Let $\mathbb{N}$ denote the set of positive natural numbers.

Definition 2. If A, B and $A_0 \subseteq A$ are sets, and a function $f : A \to B$, then $f|A_0$ denotes the function f whose domain has been restricted to A_0, such that $\{f(a) \mid a \in A_0\}$.

Definition 3. A section S_n of the positive integers, is a set that equals $\{x \in \mathbb{N} \mid x < n\}$. A section S_{n+1} denotes the set $\{x \in \mathbb{N} \mid x \leq n\}$.

Definition 4. The collection of all subsets of a set X is denoted by $\mathcal{P}(X)$.

Definition 5. A relation C on a set A is called an order relation if it has the following properties:

1. (Comparability) For every x and y in A for which $x \neq y$, either xCy or yCx.

2. (Non-reflexivity) For no x in A does the relation xCx hold.

3. (Transitivity) If xCy and yCz, then xCz.

Definition 6. Given a set A a relation $\prec$ on A is called a *strict partial order* on A if it has the following properties:

1. (Non-reflexivity) The relation $a \prec a$ never holds.

2. (Transitivity) If $a \prec b$ and $b \prec c$, then $a \prec c$.

Definition 7. A relation $\preceq$ on a set A is called *partial order* relation, if it has the following properties:

1. (Reflexivity) The relation $a \preceq a$ holds for all $a \in A$.

2. (antisymmetry) If $a \preceq b$ and $b \preceq a$, then $a = b$.

3. (Transitivity) If $a \preceq b$ and $b \preceq c$, then $a \preceq c$.

Definition 8. A topology on a set X, is a collection $\mathcal{T}$ of subsets of X having the following properties:

1. $\emptyset$ and X are in $\mathcal{T}$.

2. The union of the elements of any sub-collections in $\mathcal{T}$ is in $\mathcal{T}$.

3. The intersection of the elements of any finite sub-collection of $\mathcal{T}$ is in $\mathcal{T}$.

Definition 9. A topological space X is called a *Hausdorff space* if for each pair x_1, x_2 of distinct points of X, there exists neighborhoods U_1, and U_2 of x_1 and x_2 respectively that are disjoint.

2 Set theory and logic

2.1 Fundamental Concepts

2.1.1 Exercises

Exercise 2.1. Show that

$$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D).$$

Solution 2.1. Let $(x, y) \in (A \times B) \cap (C \times D)$, then $(x, y) \in A \times B$ and $(x, y) \in C \times D$. It follows that $x \in A, C$ and $y \in B, D$, then $x \in A \cap C$ and $y \in B \cap D$, so

$$(A \times B) \cap (C \times D) \subseteq (A \cap C) \times (B \cap D).$$

For the reverse inclusion, let $(x, y) \in (A \cap C) \times (B \cap D)$, then $x \in A \cap C$ and $y \in B \cap D$. As $x \in A, C$ and $y \in B, D$, then $(x, y) \in A \times B$ and $(x, y) \in C \times D$. So

$$(A \times B) \cap (C \times D) \supseteq (A \cap C) \times (B \cap D)$$

as desired.

Exercise 2.2. Show that

$$(A \times B) \setminus (C \times D) = (A \setminus C) \times (B \setminus D)$$

Solution 2.2. If $(x, y) \in (A \times B) \setminus (C \times D)$, then $(x, y) \notin C \times D$. It is then certainly in $(A \setminus C) \times (B \setminus D)$ for x cannot be in C and y cannot be in D.

If $(x, y) \in (A \setminus C) \times (B \setminus D)$, then $x \in A \setminus C$ and $y \in B \setminus D$. It clearely follows that x cannot be in C and y cannot be in D. As $(x, y) \notin C \times D$ it must be in $(A \times B) \setminus (C \times D)$, as desired.

Exercise 2.3. Formulate and prove DeMorgan's laws for arbitary unions and intersections.

Solution 2.3. The first law we will prove is

$$X \setminus \bigcap U_i = \bigcup (X \setminus U_i).$$

If $x \in X \setminus \bigcap U_i$, then $x \notin \bigcap U_i$. It follows that $x \notin U_i$ for some i, then x must be in $\bigcup (X \setminus U_i)$. So $X \setminus \bigcap U_i \subseteq \bigcup (X \setminus U_i)$. Conversely, if $x \in \bigcup (X \setminus U_i)$, then for some i, $x \notin U_i$. It follows that x cannot be in $\bigcap U_i$ and is thus in $X \setminus \bigcap U_i$. So $X \setminus \bigcap U_i \supseteq \bigcup(X \setminus U_i)$, as desired.

The second law is

$$X \setminus \bigcup U_i = \bigcap(X \setminus U_i).$$

If $x \in X \setminus \bigcup U_i$ then $x \notin U_i$ for all i. It is then certainly in $\bigcap(X \setminus U_i)$, so $X \setminus \bigcup U_i \subseteq \bigcap(X \setminus U_i)$. Conversely, if $x \in \bigcap(X \setminus U_i)$ then x cannot be in $\bigcup U_i$, so $x \in X \setminus \bigcup U_i$, and $X \setminus \bigcup U_i \supseteq \bigcap(X \setminus U_i)$, as desired.

2.2 Functions

Lemma 2.1. *Let $f : A \to B$. If there are functions $g : B \to A$ and $h : B \to A$ such that $g(f(a)) = a$ for every $a \in A$. and $f(h(b)) = b$ for every $b \in B$, then f is bijective and $g = h = f^{-1}$.*

Proof. In order to prove that f is bijective, one must to prove that f is surjective and injective. To show that f is surjective; we proceed by contradiction. Assume that f is not surjective, then there is at least one element $b_0 \in B$ for which there is no corresponding element $a \in A$. We have a function $f \circ h : B \to A \to B$ that is defined by $(f \circ h)(b) = b$ for *all* $b \in B$, contradicting our claim that the map $f : A \to B$ is not surjective, since in that case $f(a) \neq b_0$ for all $a \in A$ and not satisfy the condition that $(f \circ h)(b) = b$ for all $b \in B$.

To show that f is injective, consider a function f such that $f(a) = f(a')$, since, by the lemma, we have an function g such that $g(f(x)) = x$ for all $x \in A$, then we apply g to the equation above and get

$$f(a) = f(a') \Rightarrow g(f(a)) = g(f(a')) \Rightarrow a = a',$$

thus meeting the definition of injectivity. f is then bijective as desired.

To prove that $f^{-1} = h = g$. Since $f : A \to B$ is bijective, such that $f : a_n \mapsto b_k$, there is a bijective function $f^{-1} : B \to A$, such that $f^{-1} : b_k \mapsto a_n$. We define a function $f \circ f^{-1} : B \to A \to B$, it follows that $f \circ f^{-1} : b_k \mapsto a_n \mapsto b_k$ so that $(f \circ f^{-1})(b) = b$ for all $b \in B$, since $(f \circ h)(b) = b$ for all $b \in B$, we conclude that $h : b_k \mapsto a_n$, and so $h = f^{-1}$.

Lets consider a function $f^{-1} \circ f : A \to B \to A$, by our previous definition of f and f^{-1} it follows that $f^{-1} \circ f : a_n \mapsto b_k \mapsto a_n$. So $(f^{-1} \circ f)(a) = a$ for all $a \in A$. Since $(g \circ f)(a) = a$ for all $a \in A$, we conclude that $g : b_k \mapsto a_n$ then $f^{-1} = g = h$ as desired. □

2.2.1 Exercises

Exercise 2.4. Let $f : A \to B$. Let $A_0 \subseteq A$ and $B_0 \subseteq B$.

1. Show that $A_0 \subseteq f^{-1}(f(A_0))$ and that equality holds if f is injective.
2. Show that $f(f^{-1}(B_0)) \subseteq B_0$ and that equality holds if f is surjective.

Solution 2.4. For the first question. Let the set $f(A_0)$ be denoted by B_A. We will show that A_0 is contained in the preimage of B_A under f. If f is not injective, there may be a map to some elements $a_n \mapsto b_k$ and $a_m \mapsto b_k$ for $b_k \in B_A$ for some $a_n \neq a_m$. If $a_n \in A_0$ then a_m may or may not be in A_0. It follows that the preimage of B_A under f is at least equal to A_0, and thus $A_0 \subseteq f^{-1}(f(A_0))$. If f injective, at most one unique element $a \in A$ has been mapped to each element $b \in B$, then the preimage of B_A under f equals A_0.

For the second question. Let the set $f^{-1}(B_0)$ be denoted by A_B. We will show that the image of A_B under f is a subset of B_0. If f is not surjective, the set B_0, is at least equal to the set $f(A_B)$, as f may or may not map an element $a \in A_B$ to every element $b \in B_0$. And so $f(f^{-1}(B_0)) \subseteq B_0$. If f is surjective, then it is a given that all $a \in A_B$ is mapped to B_0, and thus $f(f^{-1}(B_0)) = B_0$.

Exercise 2.5. Let $f : A \to B$ and let $A_i \subseteq A$ and $B_i \subseteq B$ for $i \in \{0, 1\}$. Show that the following is true:

1. $B_0 \subseteq B_1 \Rightarrow f^{-1}(B_0) \subseteq f^{-1}(B_1)$.
2. $f^{-1}(B_0 \cup B_1) = f^{-1}(B_0) \cup f^{-1}(B_1)$.
3. $f^{-1}(B_0 \cap B_1) = f^{-1}(B_0) \cap f^{-1}(B_1)$.
4. $f^{-1}(B_0 \setminus B_1) = f^{-1}(B_0) \setminus f^{-1}(B_1)$.
5. $A_0 \subseteq A_1 \Rightarrow f(A_0) \subseteq f(A_1)$.
6. $f(A_0 \cup A_1) = f(A_0) \cup A(A_1)$.
7. $f(A_0 \cap A_1) \subseteq f(A_0) \cap f(A_1)$. Show that equality holds if f is injective.
8. $f(A_0 \setminus A_1) \supseteq f(A_0) \setminus f(A_1)$. Show that equality holds if f is injective.

Solution 2.5. One. Consider a function $f : A \to B$ such that $f : a_n \mapsto b_k$, if the element $b_k \in B_1$ but not in B_0, then $f^{-1}(B_0) = \emptyset$, and the equation holds, likewise if it is in B_0, then $a_n \in f^{-1}(B_0)$, and be in $f^{-1}(B_1)$ as well, as any element of the image that is in B_0, will be in B_1. So, every element in the preimage of B_1 under f will either be an element of $f^{-1}(B_0)$ or not—there is no case in which there will be more elements in $f^{-1}(B_0)$ than in $f^{-1}(B_1)$. So the equation holds.

2.3 Relations

2.4 Principle of recursive definition

Lemma 2.2. *There exists a function $f : \{1, \dots, n\} \to C$ that satisfies* (1) *for all i in its domain.*

Proof. By induction consider a function $f : \{1, \dots, n\} \to A$, where $a_0 \in A$, defined by the equation $f(1) = a_0$. It is clear that it satisfies (1) for $n = 1$. By inductive hypothesis the lemma is true for $n - 1$. To prove it for n let there be a function $f' : \{1, \dots, n-1\} \to A$ that satisfies (1). Let f' be defined by $f'(i) = f(i)$ for $i \in \{1, \dots, n-1\}$, and f by

$$\begin{aligned} f(1) &= a_0 \\ f(n) &= \rho(f'|\{1, \dots, n-1\}) \qquad \text{for } n > 1 \end{aligned}$$

Now, since $f'(i) = f(i)$ for $i \in \{1, \dots, n-1\}$, f satisfies (1). □

Lemma 2.3. *If two functions $f : \{1, \dots, n\} \to C$ and $g : \{1, \dots, m\} \to C$ satisfy* (1)*, then $f(i) = g(i)$ for all i in both domains.*

Proof. In Lemma 2.2 we showed that there exists a function $h : \{1, \dots, n\} \to C$ that satisfy (1). Let $f : \{1, \dots, n\} \to A$ and $g : \{1, \dots, m\} \to A$ be two such functions. By contradiction, let i be the smallest number for which $g(i) \neq f(i)$, it is not 1 since $f(1) = g(1) = a_0$ by definition. So we have that $f(j) = g(j)$ for all $j < i$. Since g and f satisfy (1), it follows that $f(\{1, \dots, i-1\}) = g(\{1, \dots, i-1\})$. Then $f(i) = g(i)$ for all i in both of their domains, contrary to our choice of i. □

Theorem 2.4 (Principle of recursive definition)**.** *Let A be a set and $a_0 \in A$. Then consider a function ρ that assigns to each function f mapping a nonempty section of $\mathbb{N}$ to A, an element of A. Then there is a unique function*

$$h : \mathbb{N} \to A$$

That satisfies the equation

$$\begin{aligned} h(1) &= a_0 \\ h(n) &= \rho(h|\{1, \ldots, n-1\}) \qquad \textit{for } i > 1. \end{aligned} \tag{1}$$

Proof. Consider a function $f_n : \{1, \ldots, n\} \to C$, that satisfy (1), whose existence we have proven proven in Lemma 2.2. Let the function $h : \mathbb{N} \to C$ be defined to be the union U of all the rules that define f_n. The rules for f_n is a subset of $\{1, \ldots, n\} \times C$, then U is a subset of $\mathbb{N} \times C$, and as such can be represented as ordered pairs in the form $(i, f_n(i))$ for $i \leq n$. Now it is clear that since $h(i) = f_n(i)$, and that f satisfies (1) for all i is its domain, so does h. By Lemma 2.3, $f_n(i) = f_m(i)$ for $i \leq m, n$, it follows that there is only one element of U for each i, $h(i)$ is thus well-defined.

Since there exists a function $h : \mathbb{N} \to C$ that satisfy (1), to prove that it is unique, consider by contradiction two functions $h_a : \mathbb{N} \to C$ and $h_b : \mathbb{N} \to C$ that satisfy (1) for all $i \in \mathbb{N}$, such that $h_a(i) \neq h_b(i)$. We have already proven in Lemma 2.3 that two functions $\{1, \ldots, n\} \to C$ and $\{1, \ldots, m\} \to C$ that satisfy (1) are equal for all i in both of their domains. Since h_a and h_b have $\mathbb{N}$ as their domain and they satisfy (1), $h_a(i) = h_b(i)$ for all $i \in \mathbb{N}$, contradicting our statement that $h_a(i) \neq h_b(i)$. Then h is unique as desired. □

2.4.1 Exercises

Exercise 2.6. Let $(b_1, b_2, \ldots)$ be an infinite sequence of real numbers. The sum $\sum_{k=1}^{n} b_k$ is defined by induction as follows:

$$\begin{aligned} \sum_{k=1}^{n} b_k &= b_1 \qquad \text{for } n = 1 \\ \sum_{k=1}^{n} b_k &= \left(\sum_{k=1}^{n-1} b_k\right) + b_n \qquad \text{for } n > 1. \end{aligned}$$

Choose ρ so that Lemma 2.4 applies to define the sum rigorously.

Solution 2.6. We wish to to apply Theorem 2.4 to define a function $h : \mathbb{N} \to \mathbb{R}$ rigorously such that $h(n) = \sum_{k=1}^{n} b_k$. To apply this Theorem, let ρ be defined by the equation $\rho(f) = f(m) + b_{m+1}$, where $f : \{1, \ldots, m\} \to \mathbb{R}$. Then there exists a function $h : \mathbb{N} \to \mathbb{R}$ such that

$$\begin{aligned} h(1) &= b_1 \\ h(n) &= \rho(h|\{1, \ldots, i-1\}) \qquad \text{for } i > 1 \end{aligned}$$

So this means that $h(1) = b_1$ and $h(i) = h(i-1) + b_i$ for $i > 1$. If we denote $h(n)$ by $\sum_{k=1}^{n} b_k$, we have

$$\begin{aligned}
\sum_{k=1}^{1} b_k &= b_1 \\
\sum_{k=1}^{n} b_k &= \left(\sum_{k=1}^{n-1} b_k\right) + b_n \qquad \text{for } n > 1.
\end{aligned}$$

as desired.

Exercise 2.7. Let $(b_1, b_2, \ldots)$ be an infinite sequence of real numbers. The product $\prod_{k=1}^{n} b_k$ is defined by the equations

$$\begin{aligned}
\prod_{k=1}^{1} b_k &= b_1 \qquad \text{for } n = 1 \\
\prod_{k=1}^{n} b_k &= \left(\prod_{k=1}^{n-1} b_k\right) \cdot b_n \qquad \text{for } n > 1
\end{aligned}$$

Use Theorem 2.4 to define this product rigorously.

Solution 2.7. We wish to apply Theorem 2.4 to define a function $h : \mathbb{N} \to \mathbb{R}$ rigorously so that $h(i) = \prod_{k=1}^{i} b_k$. To apply the Theorem, let ρ be defined by the equation $\rho(f) = f(m) \cdot b_{m+1}$, where $f : \{1, \ldots, m\} \to \mathbb{R}$. Then there exists a unique function

$$\begin{aligned}
h(1) &= b_1 \\
h(n) &= \rho(h|\{1, \ldots, n-1\}) \qquad \text{for } n > 1.
\end{aligned}$$

Then we have $h(1) = b_1$ and $h(n) = h(n-1) \cdot b_n$ for $n > 1$. Let $h(n)$ be denoted by $\prod_{k=1}^{n}$, then we have

$$\begin{aligned}
\prod_{k=1}^{1} b_k &= b_1 \qquad \text{for } n = 1 \\
\prod_{k=1}^{n} b_k &= \left(\prod_{k=1}^{n-1} b_k\right) \cdot b_n \qquad \text{for } n > 1
\end{aligned}$$

as desired.

Exercise 2.8. Obtain a definition of $n!$ for $n \in \mathbb{N}$ as special case of Exercise 2.7.

Solution 2.8. We wish to define $h(n) = \prod_{k=1}^{n} k = n!$ as a special case of exercise 2.7. We restrict the co-domain of h to $\mathbb{N}$. If we let b_{m+1} be denoted by $m+1$, b_1 by 1, and b_k by k, we can denote $h(n)$ by $n!$ since

$$1! = \prod_{k=1}^{1} k = 1 \qquad \text{for } n = 1$$
$$n! = \prod_{k=1}^{n} k = \left(\prod_{k=1}^{n-1} k\right) \cdot n \qquad \text{for } n > 1$$

as desired.

Exercise 2.9. Obtain a definition of a^n as a special case of exercise 2.7.

Solution 2.9. By applying the definition of $\prod_{k=1}^{n} b_k$ we got in Exercise 2.7 we have $h(n) = h(n-1) \cdot a$, by letting $a = a_k = b_k$. And if we denote $h(i)$ by a^i we have

$$a^1 = \prod_{k=1}^{1} a_k = a \qquad \text{for } n = 1$$
$$a^n = \prod_{k=1}^{n} a_k = \left(\prod_{k=1}^{n-1} a_k\right) \cdot a = a^{n-1} \cdot a \qquad \text{for } n > 1$$

as desired.

3 Topology and Continuous Functions

3.1 Topological spaces

3.1.1 Exercises

Exercise 3.1. Let X be a topological space; let A be a subset of X. Suppose that for each $x \in A$ there is an open set U containing x such that $U \subseteq A$. Show that A is open in X.

Solution 3.1. Let $\{U_\alpha\}_{\alpha\in J}$ denote the indexed family of open subsets of A. Since each element x in U_α is in A, it follows that A is open since

$$A = \bigcup_{\alpha\in J} U_\alpha. \tag{2}$$

Exercise 3.2. Let X be a set; let $\mathcal{T}_c$ be a collection of all subsets U of X such that $X \setminus U$ is either is countable or all of X. Show that $\mathcal{T}_c$ is topology.

Solution 3.2. Let $\{U_\alpha\}_{\alpha\in J}$ in all the elements of $\mathcal{T}_c$ such that $X \setminus U_\alpha$ is either countable or all of X. Since $X \setminus X = \emptyset$ is countable, X is in $\mathcal{T}_c$. Similarly, since $X \setminus \emptyset = X$, $\emptyset$ is also in $\mathcal{T}_c$. To show that $\bigcup U_\alpha$ is in $\mathcal{T}_c$; Since

$$X \setminus \bigcup U_\alpha = \bigcap (X \setminus U_\alpha) \tag{3}$$

If $X \setminus U_\alpha$ is countable, so is $\bigcap X \setminus U_\alpha$ as an intersection with a countable set. If $X \setminus U_\alpha$ not countable it equals X which is open.

To show that that $\bigcap U_\alpha$ is in $\mathcal{T}_c$; Since

$$X \setminus \bigcap U_\alpha = \bigcup (X \setminus U_\alpha) \tag{4}$$

and union of countable sets are themselves countable, $\bigcap U_\alpha$ is in $\mathcal{T}_c$. $\mathcal{T}_c$ is then a topology.

Exercise 3.3. Is the collection

$$\mathcal{T}_\infty = \{U \mid X \setminus U \text{ is infinite or empty or all of } X\}$$

a topology on a set X?

Solution 3.3. Let there be two infinite subsets of $\mathbb{Z}$ such that $U_1 = \{x \in \mathbb{Z} \mid x \geq 10\}$ and $U_2 = \{x \in \mathbb{Z} \mid x \leq 15\}$. One checks that $U_1 \cap U_2$, is not open, since it's neither infinite, empty or all of X. And thus does not meet the requirement that an arbitrary intersection of a subset U is open. So $\mathcal{T}_\infty$ is not a topology.

Exercise 3.4. If $\{\mathcal{T}_\alpha\}$ is a family of topologies on X, show that $\bigcap \mathcal{T}_\alpha$ is a topology on X.

Solution 3.4. Each topology $\mathcal{T}_\alpha$ contains X and $\emptyset$ by definition, which means that they are open in $\bigcap \mathcal{T}_\alpha$. Suppose that U_1 and U_2 are open sets of $\mathcal{T}_1$, and that a collection $\mathcal{C}$, contains arbitary unions and finite intersections of U_1 and U_2.

Given that $U_1, U_2 \in \mathcal{T}_2$, it follows that $\mathcal{C} \subseteq \mathcal{T}_2$ as well, since $\mathcal{T}_2$ is a topology. As a consequence an intersection $\mathcal{T}_1 \cap \mathcal{T}_2$ containing U is a topology.

To show that the above holds for n intersections; proceed by induction. For $n = 1$ it holds trivially. We assume that our statement holds for $n - 1$ and prove it for n. Since

$$\mathcal{T}_1 \cap \cdots \cap \mathcal{T}_n = (\mathcal{T}_1 \cap \cdots \cap \mathcal{T}_{n-1}) \cap \mathcal{T}_n \tag{5}$$

it is true by by our previous result.

Exercise 3.5. If $\{\mathcal{T}_\alpha\}$ is a family of topologies on X, is $\bigcup \mathcal{T}_\alpha$ a topology?

Solution 3.5. It is not. Given two topologies on a set $X = \{a, b, c\}$, such that $\mathcal{T}_1 = \{X, \emptyset, \{a, b\}, \{b\}\}$ and $\mathcal{T}_2 = \{X, \emptyset, \{a, b\}, \{c\}\}$. Since

$$\mathcal{T}_1 \cup \mathcal{T}_2 = \{X, \emptyset, \{a, b\}, \{b\}, \{c\}\}$$

is *not* a topology, thus the proposition fails.

Exercise 3.6. Show that if Y is a subspace of X, and A is a subset of Y, then the topology A inherits as a subspace of Y is the same as the topology it inherits as a subspace of X.

Solution 3.6. If $\mathcal{T}$ is a topology on X, let U be a a subset of $A \cap (Y \cap V)$ such that $V \in \mathcal{T}$. Since $A \subseteq Y$ then $A \cap Y = A$, so $(A \cap Y) \cap V = A \cap V$, then $U \subseteq A \cap V$. Conversely, let a set $U \subseteq A \cap Z$ such that $Z \in \mathcal{T}$, since $A \subseteq Y$, this implies

$$U \subseteq (A \cap Y) \cap Z \Rightarrow U \subseteq A \cap (Y \cap Z)$$

as desired.

Exercise 3.7. If $\mathcal{T}$ and $\mathcal{T}'$ are topologies on X and $\mathcal{T}'$ is strictly finer than $\mathcal{T}$, what can you say about the corresponding subspace topologies on the subset Y of X?

Solution 3.7. Let $\mathcal{S}$ and $\mathcal{S}'$ denote subspace topologies on Y of X such that

$$\mathcal{S} = \{Y \cap U \mid U \in \mathcal{T}\}$$
$$\mathcal{S}' = \{Y \cap U \mid U \in \mathcal{T}'\}$$

Since $\mathcal{T} \subset \mathcal{T}'$ and $Y \subseteq X$, any set $V \in Y \cap U$ where $U \in \mathcal{T}$, is in $Y \cap Z$ where $Z \in \mathcal{T}'$. Then $\mathcal{S} \subseteq \mathcal{S}'$. So $\mathcal{S}'$ is finer than $\mathcal{S}$.

Exercise 3.8. Show that the functions $\pi_1 : X \times Y \to X$ such that $\pi(x, y) = x$, and $\pi_2 : X \times Y \to Y$ such that $\pi_2(x, y) = y$ are open maps.

Solution 3.8. Let A be an open subset of X and B an open subset of Y. Then $A \times B$ is open in $X \times Y$, then $\pi_1(\{x \mid x \in A\}, \{y \mid y \in B\}) = A$ which is open in X, as desired. Likewise with $\pi_2(\{x \mid x \in A\}, \{y \mid y \in B\}) = B$, and B is open in Y, as desired.

Exercise 3.9. Show that the countable collection

$$\{\{x \mid a < x < b\} \times \{x \mid c < x < d\} \mid a, b, c, d \in \mathbb{Q}\}$$

is a basis for $\mathbb{R}^2$.

Solution 3.9. As per the first condition of a basis, all $x \in \mathbb{R}$ must lie in an open interval of $\mathbb{Q}$. To show that it does, let $i \in \mathbb{R} \setminus \mathbb{Q}$. Since $\mathbb{Q}$ is infinite, for all i there are some rational $a, b \in \mathbb{R}$ such that $a < i < b$.

To show that the intersection of two basis elements belongs to the basis. Let $U_1 \times V_1$ and $U_2 \times V_2$ belong to the basis such that

$$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2)$$

The equation shows that the intersection of two basis elements is, in fact, a basis element, and therefore belongs to the basis. As desired.

3.2 Closed Sets and Limit Points

Theorem 3.1. *(a) A simply ordered set is a Hausdorff space in the order topology.*

(b) The product of two Hausdorff spaces is a Hausdorff space.

(c) A subspace of a Hausdorff space is a Hausdorff space.

Proof. (a): A topology X in the order topology is generated by the basis $\mathcal{B}$ so that each distinct points $x, y \in X$ belongs to a basis element such that $x \in \,]a, b[$ and $y \in \,]c, d[$ for some $b \leq c$ for $x < y$, or for some $a \geq d$ if $x > y$. In either case, the neighborhood of x is disjoint from the the neighborhood of y.

(b): Let A and B be Hausdorff spaces, then for each distinct pair of $x_1, x_2 \in A$ and $y_1, y_2 \in B$, there exists a disjoint neighborhood in A and B respectively containing those points. It follows that for each $(x_1, y_1), (x_2, y_2) \in A \times B$ exists a disjoint neighborhood for each pair of first and second coordinates, as desired.

(c): Let $(X, \mathcal{T})$ denote a Hausdorff space; let $(X, \mathcal{T}_Y)$ denote the subspace of the set $Y \subseteq X$ on $\mathcal{T}$. By definition $\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}$, as a consequence, $x \in Y \cap U$ if, and only if $x \in U$. As all $U \in \mathcal{T}$ are Hausdorff, so is $Y \cap U$, as desired. □

3.2.1 Exercises

Exercise 3.10. Let $\mathcal{C}$ be a collection of subsets of X. Suppose that $\emptyset$ and X are in $\mathcal{C}$, and that finite unions and arbitary intersections of elements of $\mathcal{C}$ are in $\mathcal{C}$. Show that the the collection

$$\mathcal{T} = \{X \setminus C \mid C \in \mathcal{C}\}$$

is a topology on X.

Solution 3.10. Since $\emptyset$ and X are in $\mathcal{C}$, they are in $\mathcal{T}$ as they are each others complement.

As arbitary intersections of the elements of $\mathcal{C}$ are in $\mathcal{C}$, we can apply DeMorgan's laws to get

$$X \setminus \bigcap_{i \in J} C_i = \bigcup_{i \in J} (X \setminus C_i)$$

Likewise, as finite unions of elements of $\mathcal{C}$ are in $\mathcal{C}$ we get

$$X \setminus \bigcup_{i=1}^{n} C_i = \bigcap_{i=1}^{n} (X \setminus C_i)$$

Thus $\mathcal{T}$ is a topology by Definition 8.

Exercise 3.11. Show that if A is closed in X and B is closed in Y, then $A \times B$ is closed in $X \times Y$.

Solution 3.11. If A and B are closed in X and Y respectivly, then $X \setminus A$ and $Y \setminus B$ are open. It follows that $(X \setminus A) \times (Y \setminus B)$ is open in $X \times Y$. As $(X \setminus A) \times (Y \setminus B) = (X \times Y) \setminus (A \times B)$, it is clear that $A \times B$ is closed in $X \times Y$.

Exercise 3.12. Let X be an ordered set in the order topology. Show that $\overline{(a, b)} \subseteq [a, b]$. Under what condition does equality hold?

Solution 3.12. Since (a, b) is an open interval, it is by definition contained in the closed set $[a, b]$, so so is $\overline{(a, b)}$. $[a, b]$ is closed because it is the complement of the open set $(-\infty, a) \cup (b, \infty)$. Equality would hold if (a, b) was closed.