1 Set theory and logic

Exercise 1. Let $f : A \to B$. Let $A_0 \subseteq A$ and $B_0 \subseteq B$.

Show that $A_0 \subseteq f^{-1}(f(A_0))$ and that equality holds if f is injective.

Show that equality $f(f^{-1}(B_0)) \subseteq B_0$ holds if f is surjective.

If $f(a) \in B$ then $f(A_0) \subseteq B$, by definition $f^{-1}(B) = \{a \in A \mid f(a) \in B\}$ is the preimage of B under f, then $f^{-1}(f(A_0)) = \{a \in A \mid f(a) \in f(A_0)\}$ is the preimage of $f(A_0)$ under f. Since f is injective, there is a unique $f(a) \in B$ for every $a \in A_0$, and so $A_0 = f^{-1}(f(A_0))$.

Since f is surjective, there is at least one image of $f(a)$. The preimage image of $f(f^{-1}(B_0))$ is given by the set $\{f(a) \mid \exists a, a \in f^{-1}(B_0)\}$. And thus there exists an a such that $a \in f^{-1}(B_0)$, $B_0 = f(f^{-1}(B_0))$.

Exercise 2. Let $f : A \to B$ and let $A_i \subseteq A$ and $B_i \subseteq B$ for $i = 0 \wedge i = 1$. Show that f^{-1} preserves inclusions, unions, intersections, and differences of sets.

Since $f^{-1}(B_1)$ is the set of all values $a \in A$ such that $f(a) \in B_1$, it includes all the values of values $a \in A$ such that $f(a) \in B_0$ since $B_0 \subseteq B_1$.